Non-Commutative Rings and their Applications VIII

Lens, France

August 28 - 31, 2023

Mohamed F. Yousif

The Ohio State University

yousif.1@osu.edu

Direct Complements Almost Unique

Joint work with Yasser Ibrahim of both Tebah and Cairo Universities

Preliminaries

Definition 1 *A right R-module M is said to satisfy the (full) exchange property if for any two direct sum decompositions $M \oplus N = \oplus_{i \in I} N_i$, there exist submodules $K_i \subseteq N_i$ such that $M \oplus N = M \oplus (\oplus_{i \in I} K_i)$. If this holds only for $|I| < \infty$, then M is said to satisfy the finite exchange property.*

It is an open question due to Crawley and Jónsson whether the finite exchange property always implies the full exchange property.

A ring R is called exchange if R_R is exchange as a right R-module. Exchange rings are closely related to another interesting class of rings called clean rings that was first introduced by K. Nicholson, where a ring R is called clean if every element is the sum of an idempotent and a unit. A module M_R is called clean if $End(M_R)$ is a clean ring.

Definition 2 *A module M is called (summand-)square-free if it contains no non-zero isomorphic (summand) submodules A and B with $A \cap B = 0$.*

Definition 3 *A module M is called (summand)-dual-square-free if M has no proper (summand) submodules A and B with $M = A + B$ and $M/A \cong M/B$.*

Definition 4 *Two direct-summands A and B of a right R-module M are called perspective summands, and denoted by $A \sim B$, if they have a common complement, i.e. there exists a direct-summand $C \subseteq M$ such that $M = A \oplus C = B \oplus C$. The module M is called perspective if every two isomorphic direct-summands of M are perspective.*

The notion of perspectivity is left-right symmetric and lies strictly between the internal cancellation and stable range 1. If M is a module with the finite exchange property, then the three notions are equivalent.

Definition 5 *Ibrahim & Y (2022): Two direct-summands A and B of a right R-module M are called strongly-perspective (s-perspective for short), and denoted by $A \overset{s}{\sim} B$, if every complement of A is a complement of B and vice-versa, i.e. if for any $X \subseteq M$, $M = A \oplus X$ iff $M = B \oplus X$. A module M is called s-perspective if every two isomorphic direct-summands of M are s-perspective, and a ring R will be called s-perspective if R as a right R-module is s-perspective.*

If M has the finite exchange property and $S := End(M_R)$, then M is s-perspective iff S is right (left) quasi-duo, iff S is right (left) summand-dual-square-free.

Theorem 6 *Let M be an s-perspective module. Then M has the finite exchange property iff M is clean, iff M has the full exchange property.*

Theorem 7 *Let M be a module with the finite exchange property. If M is either summand-square-free or dual-summand-square-free, then M is clean, s-perspective and has the full exchange property. Moreover, in this case, M has the substitution and cancellation properties, and its endomorphism ring is right (left) quasi-duo and has (square) stable range* 1.

Direct Complements Unique

Definition 8 *Given a module M. A direct summand A of M is said to have a unique direct complement if whenever $M = A \oplus B = A \oplus C$, then $B = C$. The module M is said to have unique direct complements if every direct summand of M has a unique direct complement.*

Recall that, if $A \subseteq M$, then A is said to be fully invariant in M, if $f(A) \subseteq A$ for every $f \in End(M)$. A module M is called abelian if, $S =: End(M)$; i.e. idempotents of S are central.

Proposition 9 *The following conditions on a right R-modules M are equivalent:*

1. M *is abelian;*

2. *Direct summands of* M *are fully-invariant;*

3. *Direct complements of* M *are unique;*

4. *If* $M = A \oplus B$*, then* $Hom(A, B) = 0$*.*

Direct Complements Essentially Unique

Definition 10 *Given a module M. A direct summand A of M is said to have an essentially unique direct complement if whenever $M = A \oplus B = A \oplus C$, then $B \cap C$ is essential in B (also $B \cap C$ is essential in C, by symmetry). The module M is said to have essentially unique direct complements if every direct summand of M has an essentially unique direct complement.*

It is not difficult to see that direct complements of a square-free module are essentially unique.

The following notation will be used below $\Delta(M) := \{f \in S : \ker f \subseteq^{ess} M\}$, where $S = End(M_R)$.

Proposition 11 *The following conditions on a right R-modules M are equivalent:*

1. *Direct complements of M are essentially unique;*

2. *Idempotents in $End(M_R)$ are central modulo $\Delta(M)$;*

3. *Idempotents of $End(M_R)$ commute modulo $\Delta(M)$;*

4. *If $M = A \oplus B$ and $f : A \longrightarrow B$ is a homomorphism, then* $\ker f \subseteq^{ess} A$*.*

5. *If $M = A \oplus C = B \oplus D$ with $A \cong B$, then $C \cap D \subseteq^{ess} C$ (and $C \cap D \subseteq^{ess} D$).*

Remark 12 *Clearly if direct complements of M are essentially unique and $\Delta(M) \subseteq J(S)$, then M is s-perspective. The converse holds if $J(S) \subseteq \Delta(M)$, where $S := End(M_R)$.*

Example 13 *Let* $R := \mathbb{F}_2[x_1, x_2, \ldots]\langle e, f\rangle$, *subject to the following relations* $e^2 = e$, $f^2 = f$, $ex_1 = fx_1 = 0$, $efx_2 = fex_2 = 0$, $efex_3 = fefx_3 = 0, \ldots$. *It was shown by the authors that* R *is a right square-free ring, and so right direct complements of* R *are essentially unique. Moreover, if* $(x_1, x_2, \ldots)$ *is the ideal generated by* $x_1, x_2, \ldots$, *then clearly* $\overline{R} := R/(x_1, x_2, \ldots) \cong \mathbb{F}_2\langle e, f : e^2 = e, f^2 = f\rangle$. *Inasmuch as* $\overline{e}$ *and* $\overline{f}$ *do not commute and* $J(R) \subseteq (x_1, x_2, \ldots)$, *we infer that* e *and* f *don't commute modulo* $J(R)$, *and so* R *is not* s*-perspective.*

Example 14 *Let* F *be a field and* $R = F\langle x, y, z\rangle$, *subject to the relations*

$$x^2 = x,\ y^2 = y,\ xy = y,\ yx = x,\ yz = xz,\ z^2 = 0,\ zxz = 0$$

R_R *is summand-square-free and has the finite exchange property, but does not have essentially unique direct complements. However,* R *is* s*-perspective.*

Proposition 15 *If direct complements of M are essentially unique, then M is summand-square-free.*

Corollary 16 *If the direct complements of M are essentially unique, then M is Dedekind-finite.*

Corollary 17 *If the right direct complements are essentially unique in a ring R, then R is left and right summand-dual-square-free.*

Corollary 18 *Let M be a module whose direct complements are essentially unique. If M has the finite exchange, then M is clean, s-perspective and has the full exchange.*

Direct Complements Almost Unique

In this section we introduce and investigate a dualization of the notion of direct complements are essentially unique. But first, a submodule N of M is called small in M if, $M = N + L$ implies $M = L$.

Definition 19 *A direct summand A of a module M is said to have an almost unique direct complement if, whenever $M = A \oplus B = A \oplus C$, then $(B + C)/B \ll M/B$ (also $(B + C)/C \ll M/C$, by symmetry). Direct complements in a module M are said to be almost unique if every direct summand A of M has an almost unique direct complement.*

The following notation will be used below $\nabla(M) := \{f \in S : Imf \ll M\}$, where $S = End(M_R)$.

Proposition 20 *For a right R-module M, the following conditions are equivalent:*

1. *Direct complements of M are almost unique;*

2. *Idempotents in $End(M_R)$ are central modulo $\nabla(M)$;*

3. *Idempotents of $End(M_R)$ commute modulo $\nabla(M)$.*

4. *If $M = A \oplus B$ and $f : A \longrightarrow B$ is a homomorphism, then $\operatorname{Im} f \ll B$;*

5. *If $M = A \oplus B = A \oplus C$, then $A \cap (B + C) \ll A$;*

6. *If $M = A \oplus B = A \oplus C$, then $A \cap (B + C) \ll M$;*

7. *If $M = A \oplus C = B \oplus D$ with $A \cong B$, then $(C + D)/C \ll M/C$ (also $(C + D)/D \ll M/D$, by symmetry).*

A module M is said to be N-epi-projective if, for any epimorphisms $f : N \to L$ and $g : M \to L$, there exists a homomorphism $h : M \to N$ such that $g = f \circ h$. M is said to be epi-projective if it is M-epi-projective. It is known that if M is epi-projective then $J(S) = \nabla(M)$, where $S = End(M_R)$.

Theorem 21 *The following conditions on an epi-projective module M are equivalent:*

1. *Direct complements of M are almost unique;*
2. *Idempotents in $S := End(M_R)$ are central modulo $J(S)$;*
3. *Idempotents of $S := End(M_R)$ commute modulo $J(S)$;*
4. *$S := End(M_R)$ is s-perspective;*
5. *M is s-perspective.*

While the notion of "direct complements essentially unique" is not left-right symmetric, the dual notion of "direct complements almost unique" is left-right symmetric.

Corollary 22 *A ring R is s-perspective iff direct complements of R are almost unique. In particular, the notion of "Direct Complements Almost Unique" is left-right symmetric for rings.*

Example 23 *Let R be a unit-regular ring and $S := \mathbb{M}_2(R)$. Then S is a perspective ring whose direct complements are not almost unique.*

The next result is a dualization of the above noted fact that direct complements of square-free modules are essentially unique.

Proposition 24 *If M is a dual-square-free module, then direct complements of M are almost unique.*

Example 25 *If R is the free algebra $\mathbb{Q}\langle x, y\rangle$, then clearly R is an abelian ring whose direct complements are almost unique. However, R is not quasi-duo (dual-square-free).*

Proposition 26 *If the direct complements of* M *are almost unique, then* M *is summand-dual-square-free.*

Corollary 27 *If the direct complements of* M *are almost unique, then* M *is Dedekind-finite.*

Remark 28 *We have examples that show the notion of Direct Complements Almost Unique (*s*-perspective) lies strictly between the notions of dual-square-free and summand-dual-square-free. In the ring case, while we still don't know if the notion of quasi-duo (dual-square-free) is left-right symmetric, we should observe that both notions of "Direct Complements Almost Unique" and "summand-dual-square-free" are left-right symmetric.*

Theorem 29 *Let* M *be a module whose direct complements are almost unique. If* M *has the finite exchange, then* M *is clean,* s*-perspective and has the full exchange.*

Thank you